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Improved ε expansion for three-dimensional turbulence: Summation of nearest dimensional singularities

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An improved ε expansion in the d-dimensional ($d>2$) stochastic theory of turbulence is constructed by taking into account pole singularities at $d\to 2$ in coefficients of the ε expansion of universal quantities. Effectiveness of the method is illustrated by a two-loop calculation of the Kolmogorov constant in three dimensions.

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The renormalization-group (RG) method in the theory of turbulence is based on the stochastic Navier-Stokes equation with a Gaussian random force [1–3]. One of the central problems is the calculation of the Kolmogorov constant C: the dimensionless amplitude in the scaling law [4]

$$S_2(r) = C(\overline{\mathcal{E}}r)^{2/3} \tag{1}$$

expressing the dependence of the second-order structure function $S_2(r)$ on the relative distance r in the inertial range $r_d \ll r \ll L$. Here, L is the external length of turbulence, r_d the dissipative length, and $\overline{\mathcal{E}}$ the energy injection rate per unit mass (which, in the steady state, coincides with the dissipation rate).

There have been several attempts to solve this problem [5–14], but they all suffer from ambiguities in connecting model parameters and observable quantities. Due to this, there are significant discrepancies in the predicted numerical values for C (the spread is about a factor of 2). In this Rapid Communication we analyze reasons of this unsatisfactory situation and present results of a calculation based both on an expression of C in terms of universal quantities and account of additional singularities arising in two dimensions. Rather unexpectedly, the analysis reveals that these singularities have a major effect on the numerical values of observable quantities well above two dimensions. We show that a partial summation of these singularities is possible and significantly improves the numerical value obtained for C. To assess properties of the expansion produced within the RG approach, we have carried out the calculation in the two-loop approximation (the results of Refs. [5–14] were obtained in the one-loop approximation).

It should be emphasized that the RG yields for the structure functions a representation in which the Kolmogorov power of r is multiplied by an unknown function $\varphi(z)$ of the ratio $z=r/L$. In the inertial range $z \ll 1$, therefore to find the behavior of the structure functions in this range it is necessary to know the asymptotics of $\varphi(z)$ for $z\to 0$. The behavior $\varphi(z)\sim$const corresponds to the Kolmogorov scaling, whereas anomalous scaling means that $\varphi(z)\propto z^a$ with $a<0$. It is not possible to determine the scaling function φ from the RG equations and more sophisticated methods like infrared perturbation theory and short-distance expansion with the use of renormalization of composite operators have been used to this end [6], but without clear-cut conclusions yet. Experimentally the anomalous-scaling exponent in the function S_2 is at least small, if not zero [15]. In the third-order structure function S_3—which we also use in our calculation—there is no anomalous scaling at all. Therefore we think that the factor C in Eq. (1) may be consistently calculated as a constant independent of r.

In the RG approach to d-dimensional turbulence a powerlike correlation function of the random force is often used: $\langle ff\rangle \sim D_0 k^{4-d-2\varepsilon} \equiv d_f(k)$. In the RG framework various quantities may then be calculated in the form of an ε expansion which subsequently must be extrapolated to the physical value $\varepsilon=2$. For some important quantities the ε expansion breaks off, which for the function $S_2(r)$ yields the Kolmogorov exponent $\frac{2}{3}$ [as in Eq. (1)] at $\varepsilon=2$. To find the Kolmogorov constant the amplitude of this function has to be calculated, which, however, can be done only approximately, because its ε expansion does not break off. In calculation of the amplitude, apart from technical difficulties, a principal problem arises as well: the answer for $S_2(r)$ has to be expressed in terms of the energy injection rate $\overline{\mathcal{E}}$ [as in Eq. (1)] instead of the parameter D_0 of the powerlike forcing function. Different ways to treat this problem in Refs. [5–14] have led to different one-loop values of C.

In Ref. [7] (see also Refs. [5,6]) the connection between D_0 and $\overline{\mathcal{E}}$ was sought with the aid of the exact relation

$$\overline{\mathcal{E}} = \frac{(d-1)}{2(2\pi)^d}\int d\mathbf{k}\, d_f(k). \tag{2}$$

In the unphysical region $\varepsilon<2$ this integral has to be cut off at wave numbers of the order of $\Lambda \equiv r_d^{-1}$. At fixed $\overline{\mathcal{E}}$ this procedure introduces, first, dependence on ε of the form $D_0 \sim (2-\varepsilon)$ in D_0 (which has to be taken into account in the construction of the ε expansion), and second, an ambiguity connected with the possibility to replace the upper limit Λ by $c\Lambda$ with an arbitrary coefficient c. The first feature is rather natural, because the powerlike forcing $d_f(k)\sim(2-\varepsilon)k^{4-d-2\varepsilon}$ reproduces in the limit $\varepsilon\to 2$ the realistic forcing by infinitely large eddies: $d_f(k)\sim\delta(\mathbf{k})$. The second feature, however, introduces arbitrariness in the sought connec-

tion between D_0 and $\overline{\mathcal{E}}$ through the coefficient $c^{2\varepsilon-4}$, which in turn renders the ε expansion of D_0-dependent quantities ambiguous (in Ref. [7] the simplest choice $c=1$ was used). This is a reflection of the fact that the physical content of the theory remains unaltered when D_0 is multiplied by an arbitrary function $F(\varepsilon)$ with $F(2)=1$.

Another way to fix the connection between D_0 and $\overline{\mathcal{E}}$ has been used in Refs. [8–14]. It amounts to the use of an exact relation (for the physical value $\varepsilon=2$ of the falloff exponent) which allows to connect the spectral energy flux with an integral of a third-order correlation function, the latter being subsequently calculated in the form of an ε expansion. The use of this relation in the unphysical region $\varepsilon<2$ is tantamount to a certain choice of the function $F(\varepsilon)$ mentioned above.

Thus the ε expansion of the Kolmogorov constant in the model with the powerlike forcing is not unambiguous. Therefore a better or worse agreement with the experimental value of C at the one-loop level does not bear much meaning until a procedure for subsequent approximations has been pointed out and the stability of obtained results checked. On the other hand, since the real value of the expansion parameter $\varepsilon=2$ is not small, it is difficult to expect good quantitative results without estimating—at least approximately—higher orders of the ε expansion.

In the model at hand, only quantities independent of D_0 have rigorously unambiguous dependence on ε (we will call them universal). Such quantities are, e.g., critical exponents and dimensionless ratios of structure functions $S_n(r)$, the skewness factor $\mathcal{S}\equiv S_3/S_2^{3/2}$ in particular. Calculation of universal quantities with the use of the RG method and the ε expansion yields unambiguous results and cannot lead to such ''paradoxes'' as different one-loop values for the Kolmogorov constant.

In view of this we have pursued the goal of finding a suitable universal quantity the physical value of which would be simply connected with the Kolmogorov constant, and calculating this quantity with the aid of the RG. The skewness factor $\mathcal{S}$, connected with the Kolmogorov constant through the exact relation $C=(-4/5\mathcal{S})^{2/3}$ [4,16], might serve as such a quantity. However, in the unphysical region $\varepsilon<2$ the structure function $S_2(r)$ in the model with the powerlike forcing correlation $d_f(k)\sim k^{4-d-2\varepsilon}$ contains—at $\varepsilon<\frac{3}{2}$—an independent of r UV-divergent additive term $\sim\Lambda^{2-4\varepsilon/3}$ [for $S_3(r)$ this problem is absent, see below]. As a consequence, a straightforward generalization of the skewness factor $\mathcal{S}=S_3/S_2^{3/2}$ to the region $\varepsilon<\frac{3}{2}$ becomes pointless, because the powers of r in this definition do not cancel due to the constant term in $S_2(r)$. Therefore as the desired universal quantity we chose the ''nearest relative'' of the skewness factor, the quantity [17]

$$Q(\varepsilon)\equiv\frac{r\partial S_2(r)/\partial r}{|S_3(r)|^{2/3}}=\frac{r\partial S_2(r)/\partial r}{[-S_3(r)]^{2/3}},\tag{3}$$

independent of r in the whole region $0<\varepsilon<2$ which allows us to find the physical values of $\mathcal{S}$ and C as

$$\mathcal{S}=-\left[\frac{2}{3Q(2)}\right]^{3/2},\quad C=\frac{3}{2}\left[\frac{12}{d(d+2)}\right]^{2/3}Q(2).\tag{4}$$

The use of $S_3(r)$ in constructing universal quantities is advantageous because it can be found exactly from the spectral energy balance for all $\varepsilon<2$ in the form

$$S_3(r)=-\frac{3(d-1)\Gamma(2-\varepsilon)(r/2)^{2\varepsilon-3}D_0}{(4\pi)^{d/2}\Gamma(d/2+\varepsilon)}$$

allowing us to avoid calculation of graphs in construction of the ε expansion for $S_3(r)$. It also confirms that passing to the physical limit $\varepsilon\to2$, in which $\Gamma(2-\varepsilon)\sim1/(2-\varepsilon)$, requires $D_0\simeq a(2-\varepsilon)$ to yield a finite value of $S_3(r)$. The choice of a consistent with Eq. (2) leads to the ''$\frac{4}{5}$ law'' of Kolmogorov: $S_3(r)=-\frac{4}{5}\overline{\mathcal{E}}r$ [4,16].

In the ε expansion $Q(\varepsilon)$ has the structure [17,18]

$$Q(\varepsilon)=\varepsilon^{1/3}\sum_{k=0}^{\infty}Q_k(d)\varepsilon^k\tag{5}$$

in $d>2$. The RG method allows us to find successively the coefficients of $Q_k(d)$ as a result of calculation of renormalization constants and scaling functions in perturbation theory (loop expansion). In Refs. [5–14] only the one-loop approximation was used in the calculation of the Kolmogorov constant. A detailed account of the method of calculation and the results of the two-loop integrals has been given in Ref. [17]. Specific results for the expansion (5) of the universal quantity $Q(\varepsilon)$ for $d=3$ may be found in Ref. [18]. The analytic expression for the one-loop contribution $Q_0(d)$ in Eq. (5) is [17]

$$Q_0(d)=\frac{1}{3}[4(d+2)]^{1/3}.\tag{6}$$

The two-loop contribution $Q_1(d)$ gives rise to integrals which may be evaluated numerically for any d. For $d=3$ the calculation of the Kolmogorov constant according to Eq. (4) yields the values $C^{(1)}=1.47$ (one-loop approximation) and $C^{(2)}=3.02$ (two-loop approximation). Although the two-loop correction is not small, the recommended experimental value of $C\approx2.0$ [4,19] lies in between the values given by the two approximations. Hardly any more could be expected in view of the fact that the value of the expansion parameter is not small. Below we show that the agreement with the experiment may be significantly improved by an approximate account of the high-order terms of the expansion (5).

Analysis of the dependence of the functions $Q_k(d)$ on the space dimension d shows that they have singularities at $d\leqslant2$. In particular, $Q_k(d)\sim\Delta^{-k}$ for $2\Delta\equiv d-2\to0$. This means that in the course of d tending to 2 the expansion (5) necessarily will become ''spoiled,'' because the relative contribution of the high-order terms will grow without limit. In the present two-loop approximation this feature shows in that the ratio $Q_1(d)/Q_0(d)$ in the limit $d\to\infty$ (far away from all singularities) is about $\frac{1}{20}$ and monotonically grows with decreasing d assuming at $d=3$ the value $\simeq\frac{1}{2}$ of which the ma-

jor part is brought about by graphs singular in the limit $2\Delta = d-2 \to 0$. This gives rise to hope that summation of leading Δ singularities in Eq. (5) allows us to improve quantitative results of the RG theory.

In the theory of turbulence the space dimension $d=2$ is exceptional from both the physical point of view (additional conservation laws, inverse energy cascade) and the formal procedure of UV renormalization, because in the limit $d \to 2$ new divergences appear in the graphs of the perturbation theory. These divergences show as poles in Δ in the coefficients $Q_n(d)$ for $n \geqslant 1$ in Eq. (5). A consistent procedure to deal with these divergences by an additional renormalization—which we shall use—has been developed and gives rise to a two-parameter (ε,Δ) expansion [20] of all renormalized quantities.

It should be noted that in the (ε,Δ) expansion the reversal of the direction of the energy cascade near two dimensions does not show. This reversal takes place on the crossover curve of zero transfer of the energy—i.e., on the borderline between the direct and inverse energy cascades—in the (d,ε) plane [21] (instead of ε the exponent of the inertial-range energy spectrum $E(k) \sim k^{-m}$ is used in Ref. [21]; $m = \frac{4}{3}\varepsilon - 1$ for $\varepsilon \leqslant 2$). The point is that the center of the (ε,Δ) expansion $\varepsilon=0$, $d=2$, the final point of extrapolation $\varepsilon=2$, $d=3$, as well as the segment of the line passing through these points all lie in the region of the direct cascade far away from its borderline. Therefore in extrapolation along this line segment the problem of the inverse cascade does not arise.

In Ref. [20] this two-parameter renormalization procedure was considered an alternative to the usual ε expansion. We exploit it in a different manner—as a way to improve the expansion (5) by carrying out an approximate summation of the high-order contributions.

To single out the leading poles, express $Q_k(d)$ as

$$Q_k(d) = \Delta^{-k} q_k(\Delta), \quad 2\Delta \equiv d-2, \tag{7}$$

with a regular function

$$q_k(\Delta) = \sum_{l=0}^{\infty} q_{kl} \Delta^l. \tag{8}$$

Substitution of the expressions from Eqs. (7) and (8) in Eq. (5) leads for the quantity Q to the representation

$$Q(\varepsilon) = \varepsilon^{1/3} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (\varepsilon/\Delta)^k q_{kl} \Delta^l. \tag{9}$$

The (ε,Δ) expansion corresponds to the asymptotic regime $\varepsilon \sim \Delta \to 0$, $\Delta/\varepsilon = \text{const}$. Hence the quantities $(\varepsilon/\Delta)^k$ in Eq. (9) are not considered small and the powers Δ^l play the role of a formal small parameter. The quantity Q from Eq. (9) in the nth-order approximation is

$$\varepsilon^{1/3} \sum_{k=0}^{\infty} \sum_{l=0}^{n-1} (\varepsilon/\Delta)^k q_{kl} \Delta^l \equiv Q^{(n)}_{\varepsilon,\Delta}, \quad n \geqslant 1, \tag{10}$$

which corresponds to an approximate calculation of the coefficients (7) of the ε expansion (5) with the account of n terms in the sum (8). For a RG calculation of the quantity $Q^{(n)}_{\varepsilon,\Delta}$ in the (ε,Δ)-expansion scheme [20] an n-loop approximation would be needed.

Let us assume for the moment that we have carried out an n-loop calculation in the usual ε expansion thus determining the following partial sum of the series (5),

$$\varepsilon^{1/3} \sum_{k=0}^{n-1} Q_k(d) \varepsilon^k \equiv Q^{(n)}_{\varepsilon}, \tag{11}$$

and an n-loop calculation in the (ε,Δ)-expansion scheme as well, hence having determined the quantity $Q^{(n)}_{\varepsilon,\Delta}$ of Eq. (10). Then we may amend the sum (11) by an approximate contribution of all higher powers of ε^k not taken into account in Eq. (11). The required information of this contribution is contained in the quantity $Q^{(n)}_{\varepsilon,\Delta}$. To obtain the improved value of Q we add the expressions (10) and (11), then subtract once the sum

$$\delta Q^{(n)} \equiv \varepsilon^{1/3} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} (\varepsilon/\Delta)^k q_{kl} \Delta^l$$

which enters twice in the sum of Eqs. (10) and (11). Thus we arrive at the following n-loop approximation:

$$Q^{(n)}_{eff} = Q^{(n)}_{\varepsilon} + Q^{(n)}_{\varepsilon,\Delta} - \delta Q^{(n)} \tag{12}$$

for Q. Our two-loop calculation yields the result

$$Q^{(1)}_{\varepsilon,\Delta} = 2\left[\frac{2(\varepsilon+\Delta)^2\varepsilon}{3(2\varepsilon+3\Delta)^2}\right]^{1/3},$$

$$\frac{Q^{(2)}_{\varepsilon,\Delta}}{Q^{(1)}_{\varepsilon,\Delta}} = \left[1 + \left(0.5181\varepsilon + \frac{1}{6}\Delta\right)\right] \tag{13}$$

for the quantities $Q^{(1)}_{\varepsilon,\Delta}$, $Q^{(2)}_{\varepsilon,\Delta}$ with the subsequent expressions for $\delta Q^{(1)}$, $\delta Q^{(2)}$:

$$\delta Q^{(1)} = \frac{2}{3}(2\varepsilon)^{1/3},$$

$$\frac{\delta Q^{(2)}}{\delta Q^{(1)}} = \left(1 + \frac{2\varepsilon}{9\Delta}\right)\left[1 + \left(0.5181\varepsilon + \frac{1}{6}\Delta\right)\right]. \tag{14}$$

Calculating at $d=3$ the quantity $Q^{(2)}_{\varepsilon}$ from Eq. (11) with the aid of Eq. (6) and the value $Q_1(3) \simeq 0.4748$ found in Refs. [17,18], and substituting the result together with Eqs. (13) and (14) in Eq. (12) we find Q_{eff} in first and second approximation: $Q^{(1)}_{eff} = 1.38$, $Q^{(2)}_{eff} = 1.84$. Substitution of these values in Eq. (4) at $d=3$ yields for the Kolmogorov constant and the skewness factor the values $C^{(1)}_{eff} = 1.79$, $C^{(2)}_{eff} = 2.37$, $S^{(1)}_{eff} = -0.33$, $S^{(2)}_{eff} = -0.22$.

In Table I we have compared values of the Kolmogorov constant calculated according to Eq. (4) in the first and second order of the usual ε expansion (C_ε), the double (ε,Δ)

TABLE I. One- and two-loop values of the Kolmogorov constant in the ε expansion (C_ε) and the double (ε,Δ) expansion ($C_{\varepsilon,\Delta}$); the contribution C_δ in Eq. (4) is from the correction $\delta Q^{(n)}$ in Eq. (12), and the value C_{eff} is from Eqs. (4) and (12).

n	C_ε	$C_{\varepsilon,\Delta}$	C_δ	C_{eff}
1	1.47	1.68	1.37	1.79
2	3.02	3.57	4.22	2.37

expansion ($C_{\varepsilon,\Delta}$), the contribution C_δ in Eq. (4) from the correction $\delta Q^{(n)}$ in Eq. (12) and the value C_{eff} obtained from Eqs. (4) and (12). In all the cases the recommended experimental value of the Kolmogorov constant $C\approx 2.0$ lies between the values of the first and second approximation. However, the difference between successive approximations is rather significant both in the ε expansion and in the (ε,Δ) expansion, let alone the leading terms of the ε expansion of the latter. For the improved ε expansion, i.e., for the quantity $C_{eff}=C_\varepsilon+C_{\varepsilon,\Delta}-C_\delta$ calculated according to Eqs. (12) and (4), however, this difference is about three times smaller leading to a far better agreement with the experimental data.

In conclusion, we have shown that a proper account of the ''nearest singularity'' in the coefficients of the ε expansion (5) leads to a significant improvement of the results of the two-loop RG calculation at $d=3$. We have analyzed the effect of this procedure for other d as well. It reduced significantly the relative contribution of the two-loop correction in the whole range considered $\infty > d \geqslant 2.5$. At the same time this contribution remained large at $d=2$, which might be an effect of singularities in the next exceptional dimension $d=1$. It is also possible that this is a reflection of the proximity to the zero-transfer crossover curve in dimensions close to 2.

Obviously, the proposed procedure of approximate summation of the ε expansion is applicable not only to the calculation of $Q(\varepsilon)$, but all universal quantities such as dimensions of composite operators.

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[1] D. Forster, D.R. Nelson, and M.J. Stephen, Phys. Rev. Lett. **36**, 867 (1976); Phys. Rev. A **16**, 732 (1977).
[2] C. De Dominicis and P.C. Martin, Phys. Rev. A **19**, 419 (1979).
[3] L.Ts. Adzhemyan, A.N. Vasil'ev, and Yu.M. Pis'mak, Teor. Mat. Fiz. **57**, 268 (1983).
[4] A.S. Monin and A.M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1975), Vol. 2.
[5] L.Ts. Adzhemyan, N.V. Antonov, and A.N. Vasil'ev, Usp. Fiz. Nauk **166**, 1257 (1996) [Phys. Usp. **39**, 1193 (1996)].
[6] L.Ts. Adzhemyan, N.V. Antonov, and A.N. Vasiliev, *The Field Theoretic Renormalization Group in Fully Developed Turbulence* (Gordon & Breach, London, 1999).
[7] L.Ts. Adzhemyan, N.V. Antonov, and A.N. Vasil'ev, Zh. Éksp. Teor. Fiz. **95**, 1272 (1989) [Sov. Phys. JETP **68**, 733 (1989)].
[8] V. Yakhot and S.A. Orszag, Phys. Rev. Lett. **57**, 1722 (1986); J. Sci. Comput. **1**, 3 (1986).
[9] W.P. Dannevik, V. Yakhot, and S.A. Orszag, Phys. Fluids **30**, 2021 (1987).
[10] Y. Zhou, G. Vahala, and M. Hossain, Phys. Rev. A **40**, 5865 (1989).
[11] D. Carati, Phys. Rev. A **41**, 3129 (1990); Phys. Fluids A **2**, 1854 (1990).
[12] S.H. Lam, Phys. Fluids A **4**, 1007 (1992).
[13] M.J. Giles, Phys. Fluids **6**, 595 (1994).
[14] Y. Nagano and Y. Itazu, Fluid Dyn. Res. **20**, 157 (1997).
[15] G.I. Barenblatt, A.J. Chorin, and V.M. Prostokishin, Physica D **127**, 105 (1999); R. Benzi, S. Ciliberto, C. Baudet, and G. Ruiz-Chavarria, *ibid.* **127**, 111 (1999).
[16] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, England, 1995).
[17] L.Ts. Adzhemyan, N.V. Antonov, M.V. Kompaniets, and A.N. Vasil'ev, Int. J. Mod. Phys. B **17**, 2137 (2003).
[18] L.Ts. Adzhemyan, N.V. Antonov, M.V. Kompaniets, and A.N. Vasil'ev, Acta Phys. Slov. **52**, 565 (2002).
[19] K.R. Sreenivasan, Phys. Fluids **7**, 2778 (1995).
[20] J. Honkonen and M.Yu. Nalimov, Z. Phys. B: Condens. Matter **99**, 297 (1996).
[21] J-D. Fournier and U. Frisch, Phys. Rev. A **17**, 747 (1977). See also the discussion in Sec. 2.9 of Ref. [6].